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Nonlinear Helmholtz oscillations in harbours and coupled basins

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Free and forced oscillations in a basin that is connected through a narrow canal to either the open sea or a second basin are considered on the assumption that the spatial variation of the free-surface displacement is negligible. The free-surface displacement in the canal is allowed to be finite, subject only to the restriction (in addition to that implicit in the approximation of spatial uniformity) that the canal does not run dry. The resulting model yields a Hamiltonian pair of phase-plane equations for the free oscillations, which are integrated in terms of elliptic functions on the additional assumption that the kinetic energy of the motion in the basin(s) is negligible compared with that in the canal or otherwise through an expansion in an amplitude parameter. The corresponding model for forced oscillations that are limited by radiation damping yields a generalization of Duffing's equation for an oscillator with a soft spring, the solution of which is obtained as an expansion in the amplitude of the fundamental term in a Fourier expansion. Equivalent circuits are developed for the various models.

1. Introduction

Helmholtz oscillations occur in a basin that is connected to a large body of water through a narrow canal (figure 1) or in two or more basins that are connected by one or more narrow canals (figure 2). Their natural frequency is small compared with the natural frequency (frequencies) of the closed basin(s). They are especially significant for typical harbours in consequence of their susceptability to excitation by tsunamis.

Consider, for example (figure 1), a basin of free-surface area S that is connected to the open sea through a canal of uniform breadth b, uniform depth d and length l and suppose that

$$\lambda \gg b, d, l, S^{\frac{1}{2}} \tag{1.1}$$

and

$$bl/S \equiv \epsilon \ll 1, \tag{1.2}$$

where λ is a characteristic wavelength. It follows from (1.1) that shallow-water theory is applicable and that the spatial (horizontal) variation of the free-surface displacement may be neglected, and from (1.2) that the potential energy of the motion in the canal may be neglected compared with that in the basin. The angular frequency of a gravity wave of sufficiently small amplitude then is given by

$$\omega = (gd/\mathscr{M}S)^{\frac{1}{2}}, \tag{1.3}$$

where $\mathscr{M}$ is a dimensionless measure of kinetic energy [see Miles & Lee (1975); $d/\mathscr{M}$ is analogous to the conductivity of an acoustical Helmholtz resonator (Rayleigh 1896), but note that the present problem is two-dimensional].

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It follows from the requirement $\lambda \propto (gd)^{\frac{1}{2}}/\omega \gg S^{\frac{1}{2}}$ that a necessary condition for the Helmholtz mode is $\mathscr{M} \gg 1$. The contribution of the uniform canal to $\mathscr{M}$ is l/b and dominates the contribution of the basin if $2\pi(l/b) \gg \ln(S/b^2)$, in which case (1.3) reduces to (Honda, Terada & Isitani 1908)

$$\omega = (gbd/lS)^{\frac{1}{2}} \equiv \omega_0 \quad (b \ll l \ll S/b). \tag{1.4}$$

The assumption of a narrow canal suggests that the dominant nonlinear effect (at least in so far as dissipation, which could be strongly affected by nonlinear motion at the ends of the canal, is small) for oscillations of finite amplitude should be the failure of the free-surface displacement in the canal, say z, to remain small compared with d. I consider this effect for a model in which the approximation of spatial uniformity and the restriction (1.2) are retained, but for which $|z/d| \ll 1$ is replaced by the weaker restriction $d+z > 0$ (the canal must not run dry). Of course, the approximation of spatial uniformity cannot be expected to remain valid for $d+z \ll d$, and this model therefore should be expected to be significant in the present context only for relatively moderate values of $-z/d$.

I consider first (in § 2) free oscillations for which the kinetic energy of the motion in the basin may be neglected compared with that in the canal, but for which the potential energy in the canal (which is neglected in the classical approximation described above) is retained on the hypotheses that $z/d = O(1)$ and $Z/d = O(\epsilon^{\frac{1}{2}})$, where Z is the free-surface displacement in the basin. This leads (in § 3) to a Hamiltonian pair of phase-plane equations that can be integrated in terms of elliptic functions within an error factor of $1+O(\epsilon)$.

I include the kinetic energy of the basin in § 4. The corresponding phase-plane equations for free oscillations resemble those in § 3 and are integrable; however, the solution cannot be expressed in terms of elliptic functions and must be expanded in powers of the amplitude to obtain analytical representations.

I then go on, in § 5, to discuss forced oscillations and the effects of radiation damping on the resonant response. The resulting differential equation is essentially that of Duffing for an oscillator with a soft spring, the solution of which may be obtained as a Fourier series in which the amplitudes of the harmonics are expanded in powers of the amplitude of the fundamental. Nonlinearity tilts the response curve (amplitude of fundamental *vs.* forcing frequency) to the left and renders it triple valued in a certain frequency range. Damping limits the amplitude and leads to the usual hysteresis and jump phenomena. Subharmonic resonance also is possible. I discuss these phenomena for monochromatic forcing in order to illustrate the qualitative similarity between the present model and the corresponding single-degree-of-freedom oscillator; however, it must not be overlooked that broadband forcing by a random input is the usual case for a harbour.

The model depicted in figure 1 is appropriate for a harbour or bay (Miles 1974). A model that is appropriate for coupled lakes (Honda *et al.* 1908; Neumann 1943; Defant 1961) is shown in figure 2.† I consider the corresponding nonlinear model, on the assumption that the kinetic energy in the basins is negligible compared with that in the canal, in § 6 and obtain results for free oscillations that are equivalent to those

† The similarity between oscillations in a two-basin Helmholtz oscillator and those in a U-tube (Newton 1686) has been noted by all of the above writers.

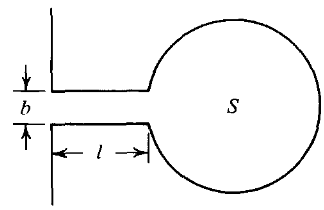

FIGURE 1. Single-basin Helmholtz oscillator.

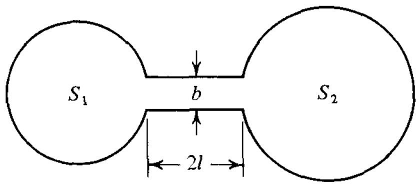

FIGURE 2. Double-basin Helmholtz oscillator.

for a single basin plus canal. This two-basin configuration evidently would be superior to its single-basin counterpart for experimental purposes.

It should perhaps be emphasized that the present nonlinearity is a consequence of the motion of the free surface in the canal and would be absent if the canal were replaced by a closed channel. It also is absent from an acoustical Helmholtz resonator.

2. Equations of motion

The shallow-water approximation to the continuity equation is

$$z_t + [(d+z)u]_x = 0, \tag{2.1}$$

where z and u are the free-surface displacement and the longitudinal velocity in the canal $(0 < x < l)$. Integrating (2.1) on the assumption that $z = z(t)$ and imposing the condition that the volumetric flux at the junction of the canal and the basin must equal the negative of the rate of change of the volume in the basin,

$$b(d+z)u = S\dot{Z} \quad (x = l), \tag{2.2}$$

where $Z = Z(t)$ is the free-surface displacement in the basin and $\dot{Z} \equiv dZ/dt$, we obtain

$$(d+z)u = (S/b)\dot{Z} + \dot{z}(l-x). \tag{2.3}$$

The kinetic energy of the fluid motion in the canal (which dominates that in the basin for $l/b \gg 1$) and the total potential energy are given by

$$T = \tfrac{1}{2}\rho b(d+z)\int_0^l u^2 dx = \tfrac{1}{2}(\rho l S^2/b)(d+z)^{-1}(\dot{Z}^2 + \epsilon\dot{Z}\dot{z} + \tfrac{1}{3}\epsilon^2\dot{z}^2) \tag{2.4}$$

and

$$V = \tfrac{1}{2}\rho g(SZ^2 + blz^2) = \tfrac{1}{2}(\rho l S^2/b)(\omega_0^2/d)(Z^2 + \epsilon z^2), \tag{2.5}$$

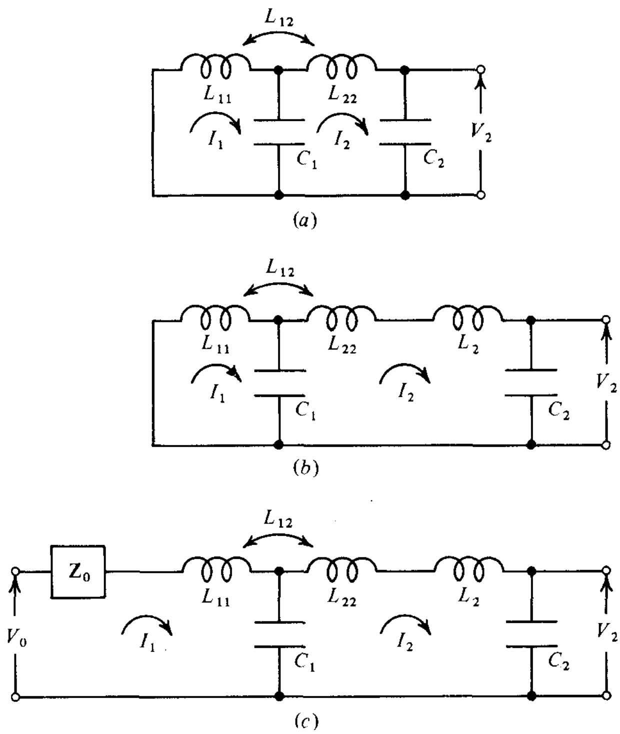

FIGURE 3. Equivalent circuits for: (*a*) §§2 and 3, (*b*) §4, (*c*) §5. The voltage across C_1 is $V_1 \equiv z$.

where ρ is the density of the fluid, ϵ is given by (1.2), and ω_0 is given by (1.4). Invoking Hamilton's principle for the Lagrangian $T - V$, we obtain the equations of motion

$$\frac{d}{dt}\left(\frac{\dot{Z}+\frac{1}{2}\epsilon\dot{z}}{d+z}\right)+\omega_0^2\frac{Z}{d} = 0 \tag{2.6a}$$

and

$$\epsilon\frac{d}{dt}\left(\frac{\frac{1}{2}\dot{Z}+\frac{1}{3}\epsilon\dot{z}}{d+z}\right)+\frac{1}{2}\left(\frac{\dot{Z}+\frac{1}{2}\epsilon\dot{z}}{d+z}\right)^2+\frac{\epsilon^2}{24}\left(\frac{\dot{z}}{d+z}\right)^2+\epsilon\omega_0^2\frac{z}{d} = 0. \tag{2.6b}$$

The kinetic and potential energies, (2.4) and (2.5), may be placed in the equivalent forms

$$T = \tfrac{1}{2}\rho g(L_{11}I_1^2+2L_{12}I_1I_2+L_2I_2^2) \tag{2.7}$$

and

$$V = \tfrac{1}{2}\rho g(C_1V_1^2+C_2V_2^2), \tag{2.8}$$

where

$$V_1 = z, \quad V_2 = Z, \quad I_1 = S\dot{Z}+bl\dot{z}, \quad I_2 = S\dot{Z} \tag{2.9}$$

are the voltages and currents in the equivalent circuit of figure 3*a*. The equivalent inductances and capacitances are given by

$$L_{11} = 2L_{12} = L_{22} = \tfrac{1}{3}\{gb(d+z)\}^{-1}l, \quad C_1 = bl, \quad C_2 = S \tag{2.10}$$

(L_{12} is the mutual inductance between L_{11} and L_{22}); see Miles (1971, 1974) and Miles & Lee (1975) for further details (but only linear problems are treated in these references). It is worth emphasizing that, although the spatial variation of the free-surface dis-

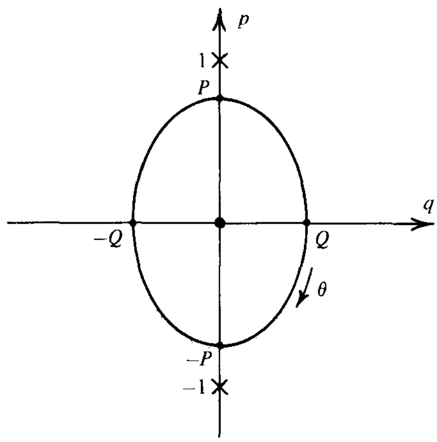

FIGURE 4. Phase plane for (3.4). The trajectories are given by $p^2 - \frac{1}{2}p^4 + q^2 = Q^2$. The crosses are saddle points.

placement has been neglected in calculating the energies, the voltage in the equivalent circuit (which is analogous to the free-surface displacement) varies continuously from zero at the input side of L_{11} (the open sea) to $V_2 \equiv Z$ at the output side of L_2 (the basin).

3. Phase-plane trajectories

The conventional approximations for the present model are to neglect z compared with d (linearization) and let $\epsilon \downarrow 0$ in (2.6a), which then yields a simple harmonic solution for Z with the frequency ω_0. The corresponding approximation to z/d may be determined from (2.6b) and is $o(\epsilon)$, $O(\epsilon)$, or $O(1)$ for $Z/d = o(\epsilon)$, $O(\epsilon)$, or $O(\epsilon^{\frac{1}{2}})$, respectively; moreover, the second term in (2.6b) is negligible compared with, of the same order of magnitude as, or dominates the first term therein in these three régimes. This, together with the form of (2.6a) suggests the introduction of the dimensionless, canonically conjugate variables p and q through the transformation

$$Z = (2\epsilon)^{\frac{1}{2}} d q(\theta), \quad \dot{Z} + \tfrac{1}{2}\epsilon \dot{z} = (2\epsilon)^{\frac{1}{2}} \omega_0 (d+z) p(\theta), \quad \theta = \omega_0 t, \qquad (3.1\,a, b, c)$$

the substitution of which into (2.6b) and (2.4) + (2.5) yields

$$z/d = -p^2 + (\epsilon/2)^{\frac{1}{2}} q + O(\epsilon) \qquad (3.2)$$

and the dimensionless Hamiltonian

$$H \equiv (T+V)/(2\rho g b d^2 l) = \tfrac{1}{2}(p^2+q^2) - \tfrac{1}{4}p^4. \qquad (3.3)$$

The corresponding canonical equations, which also may be obtained from the substitution of (3.1) into (2.6a) and the identity $\dot{Z} \equiv \omega_0 dZ/d\theta$, are

$$dq/d\theta = \partial H/\partial p = (1-p^2)p, \quad dp/d\theta = -\partial H/\partial q = -q, \qquad (3.4\,a, b)$$

where, here and subsequently, $O(\epsilon)$ error terms are implicit.

It follows from the usual phase-plane formalism (Stoker 1950) that (3.4) have a centre at $p = q = 0$ and a pair of saddle points at $p = \pm 1$ and $q = 0$; see figure 4. The phase-plane trajectories are given by the energy integral

$$H = \text{constant} \equiv \tfrac{1}{2}Q^2 \equiv (Z_0/2\epsilon^{\frac{1}{2}} d)^2, \qquad (3.5)$$

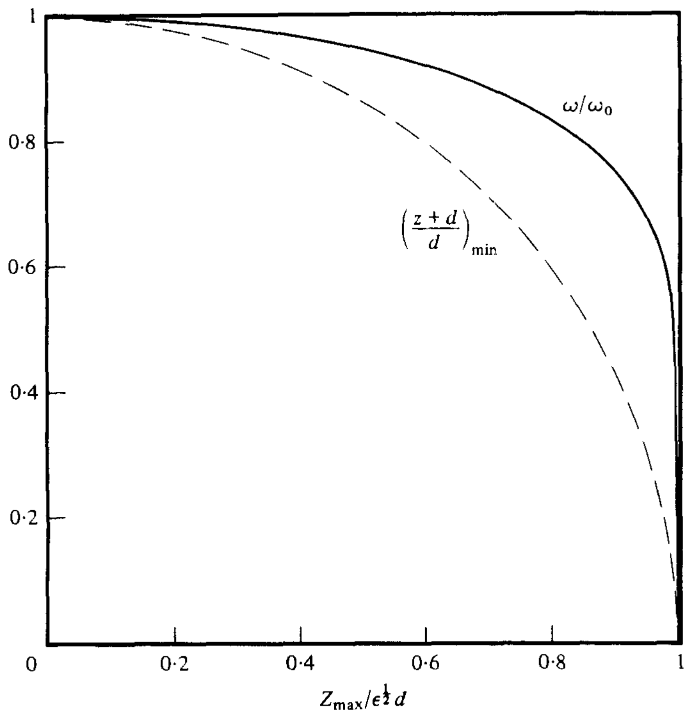

FIGURE 5. The finite-amplitude reduction factor for the natural frequency of either the single-basin or double-basin Helmholtz oscillator (——), as given by (3.9), and the ratio of the minimum depth to the quiescent depth in the canal (– – –), as given by (3.10).

where $\pm Q$ is the value of q at $p = 0$. These trajectories are closed orbits around the centre at $p = q = 0$ if and only if $Q^2 < \frac{1}{2}$ and then intersect the p axis ($q = 0$) at

$$p = \pm[1-(1-2Q^2)^{\frac{1}{2}}]^{\frac{1}{2}} \equiv \pm P. \tag{3.6}$$

The limiting trajectory (or *separatrix*), $Q^2 = \frac{1}{2}$, passes through the saddle points, which then correspond to the limiting condition $z = -d$; however, the period then is infinite (see below).

Eliminating q between (3.4*b*) and (3.5), we obtain the first-order differential equation

$$(dp/d\theta)^2 + p^2 - \tfrac{1}{2}p^4 = Q^2, \tag{3.7}$$

which may be integrated to obtain

$$p = -P\,\mathrm{sn}\,\phi, \quad q = Q\,\mathrm{cn}\,\phi\, dn\,\phi, \quad \phi = Q\theta/P, \quad k^2 = P^4/2Q^2 = P^2/(2-P^2), \tag{3.8a, b, c, d}$$

where sn, cn and dn are Jacobi elliptic functions of modulus k, and θ is measured from the point on the trajectory at which $p = 0$ and $q \equiv Q$. The corresponding period is given by (see figure 5)

$$T/T_0 \equiv \omega_0/\omega = (2P/\pi Q)K(k). \tag{3.9}$$

The minimum depth in the canal is given by (figure 5)

$$\{(z+d)/d\}_{\min} = 1 - P^2 = (1-2Q^2)^{\frac{1}{2}}. \tag{3.10}$$

Expanding (3.8) and (3.9) in powers of P (or Q), we obtain

$$p = -P\{(1+\tfrac{1}{32}P^2)\sin\omega t+\tfrac{1}{32}P^2\sin 3\omega t+O(P^4)\}, \tag{3.11a}$$

$$q = Q\{(1-\tfrac{3}{32}Q^2)\cos\omega t+\tfrac{3}{32}Q^2\cos 3\omega t+O(Q^4)\} \tag{3.11b}$$

and

$$\omega/\omega_0 = 1-\tfrac{3}{8}Q^2-\tfrac{69}{256}Q^4+O(Q^6), \tag{3.12}$$

from which it appears that the effects of nonlinearity are rather mild for moderate values of Q. The approximation (3.12) differs from (3.9) by less than 1% for $2Q < 0{\cdot}5$.

It is rather clear that the assumptions on which the preceding results are based could not remain valid as $Q^2\uparrow\frac{1}{2}$, but it is perhaps worth noting that the limiting trajectory is given by

$$p = -\tanh(\theta/\surd 2), \quad q = 2^{-\frac{1}{2}}\operatorname{sech}^2(\theta/\surd 2) \quad (Q^2=\tfrac{1}{2}). \tag{3.13a, b}$$

4. Inclusion of basin kinetic energy

The kinetic energy of the fluid motion in the basin may be posed in the form

$$T_2 = \tfrac{1}{2}\rho(\mathscr{M}_2/d)(S\dot{Z})^2 \equiv \tfrac{1}{2}(\rho l S^2/b)(\mu/d)\dot{Z}^2 = \tfrac{1}{2}\rho g L_2 I_2^2, \tag{4.1}$$

where $\mathscr{M}_2$ is the contribution of the basin to the inertial parameter introduced in (1.3), $\mu = \mathscr{M}_2 b/l$, and $L_2 = \mathscr{M}_2/gd$ (see figure 3b). Various approximations, including upper and lower bounds, to $\mathscr{M}_2$ are developed by Miles & Lee (1975).

Adding T_2 to the kinetic energy of the fluid motion in the channel, (2.4), we find that (2.6a) must be replaced by

$$\frac{d}{dt}\left(\frac{\dot{Z}+\frac{1}{2}\epsilon\dot{z}}{d+z}+\mu\frac{\dot{Z}}{d}\right)+\omega_0^2\frac{Z}{d} = 0; \tag{4.2}$$

(2.6b) remains unchanged. The dimensionless momentum p, (3.1b), therefore must be replaced by

$$\hat{p} = (2\epsilon)^{-\frac{1}{2}}\omega_0^{-1}\{(d+z)^{-1}(\dot{Z}+\tfrac{1}{2}\epsilon\dot{z})+\mu d^{-1}\dot{Z}\}, \tag{4.3}$$

whilst (3.1a, c) remain unchanged. However, we anticipate that

$$\hat{p} = (1+\mu-\mu p^2)p, \tag{4.4}$$

and find it convenient to work with p. The counterparts of (3.2)–(3.4) then are

$$z/d = -p^2+(\epsilon/2)^{\frac{1}{2}}(1+\mu-3\mu p^2)^{-1}q, \tag{4.5}$$

$$H = \tfrac{1}{2}(1+\mu)p^2+\tfrac{1}{2}q^2-(\tfrac{1}{4}+\mu)p^4+\tfrac{1}{2}\mu p^6, \tag{4.6}$$

$$dq/d\theta = \partial H/\partial\hat{p} = (1-p^2)p, \quad d\hat{p}/d\theta = -\partial H/\partial q = -q. \tag{4.7a, b}$$

If $\mu < \frac{1}{2}$, equations (4.7) have a centre at $p = q = 0$, a pair of saddle points at $p = \pm 1$ and $q = 0$, and a pair of centres at $p = \pm p_*$ and $q = 0$, where ${p_*}^2 = (1+\mu)/3\mu$. Trajectories around the latter centres are inadmissible in the present context, since $p_*^2 > 1$ (for $\mu < \frac{1}{2}$) implies $z+d < 0$. If $\mu > \frac{1}{2}$ the saddle points appear at $p = \pm p_*$ and $q = 0$, whilst the outer centres appear at $p = \pm 1$ and $q = 0$; however, the relative disposition of the saddle points and centres is unchanged, since $p_*^2 < 1$ for $\mu > \frac{1}{2}$. If $\mu = \frac{1}{2}$ the outer centres and saddle points coalesce, but the resulting singularity is qualitatively similar to a saddle point with respect to the closed trajectories $|p| < 1$ and $|q| < Q$.

Eliminating q between the energy integral $H = \frac{1}{2}Q^2$ and (4.7b), we obtain

$$q = \{Q^2 - (1+\mu)p^2 + (\tfrac{1}{2} + 2\mu)p^4 - \mu p^6\}^{\frac{1}{2}}, \tag{4.8a}$$

$$\theta = -\int_0^p (1+\mu - 3\mu x^2)\{Q^2 - (1+\mu)x^2 + (\tfrac{1}{2} + 2\mu)x^4 - \mu x^6\}^{-\frac{1}{2}} dx, \tag{4.8b}$$

and

$$\frac{\omega}{\omega_0} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} (1+\mu - 3\mu P^2 \sin^2\alpha)\,[1+\mu - (\tfrac{1}{2} + 2\mu)P^2(1+\sin^2\alpha) + \mu P^4(1+\sin^2\alpha + \sin^4\alpha)]^{-\frac{1}{2}} d\alpha, \tag{4.9}$$

where P is determined by

$$Q^2 - (1+\mu)P^2 + (\tfrac{1}{2} + 2\mu)P^4 - \mu P^6 = 0. \tag{4.10}$$

The integrals are no longer elliptic but may be evaluated either numerically or by expanding in powers of Q. The former procedure yields results that are qualitatively similar to (3.8) and (3.9). The latter procedure leads to [cf. (3.11b) and (3.12)]

$$q = Q\{(1 - \tfrac{3}{32}\hat{Q}^2)\cos\omega t + \tfrac{3}{32}\hat{Q}^2 \cos 3\omega t + O(\hat{Q}^4)\}, \quad \hat{Q} = Q/(1+\mu) \tag{4.11a, b}$$

and

$$\frac{\omega}{\omega_\mu} = 1 - \tfrac{3}{8}\hat{Q}^2 - (\tfrac{69}{256} + \tfrac{15}{16}\mu)\hat{Q}^4 + O(\hat{Q}^6), \quad \omega_\mu = \omega_0/(1+\mu)^{\frac{1}{2}}, \tag{4.12a, b}$$

where ω_μ is the natural frequency in the limit $\hat{Q} \to 0$. It is evident from (4.11) and (4.12) that the present nonlinearity is negligible if $\mu \gg 1$, which reflects the fact that the kinetic energy then is concentrated in the mouth of the basin.

5. Forced oscillations with radiation damping

External forcing by a prescribed displacement $Z_0(t)$ at the mouth of the canal may be incorporated in the equations of motion by replacing Z by $Z - Z_0$ in (2.6a) and (4.2) and z by $z - Z_0$ in (2.6b). External loading may be incorporated by replacing Z_0 by $Z_0 - \mathbf{z}_0 I_1$, where $\mathbf{z}_0$ is a radiation-impedance operator [see Miles & Lee (1975), wherein $\mathbf{z}_0 \equiv Z_E$ is a complex impedance]; the resulting equivalent circuit is shown in figure 3(c). Frictional losses may be incorporated by introducing appropriate resistors in the equivalent circuit (Miles & Lee 1975).

Introducing the dimensionless operator

$$\mathscr{X} = (gd/\omega_0)\mathbf{z}_0 \tag{5.1}$$

and

$$\beta = b/l \tag{5.2}$$

and neglecting the basin kinetic energy ($\mu \ll 1$), we obtain

$$\frac{d}{dt}\left(\frac{\dot{Z} + \frac{1}{2}\epsilon\dot{z}}{d+z}\right) + \omega_0^2\left\{\frac{Z}{d} + \beta\mathscr{X}\left(\frac{\dot{Z} + \epsilon\dot{z}}{\omega_0 d}\right)\right\} = \omega_0^2\frac{Z_0}{d}, \tag{5.3a}$$

$$\frac{d}{dt}\left(\frac{\frac{1}{2}\dot{Z} + \frac{1}{3}\epsilon\dot{z}}{d+z}\right) + \frac{1}{2\epsilon}\left(\frac{\dot{Z} + \frac{1}{2}\epsilon\dot{z}}{d+z}\right)^2 + \frac{\epsilon}{24}\left(\frac{\dot{z}}{d+z}\right)^2 + \omega_0^2\left\{\frac{z}{d} + \beta\mathscr{X}\left(\frac{\dot{Z} + \epsilon\dot{z}}{\omega_0 d}\right)\right\} = \omega_0^2\frac{Z_0}{d} \tag{5.3b}$$

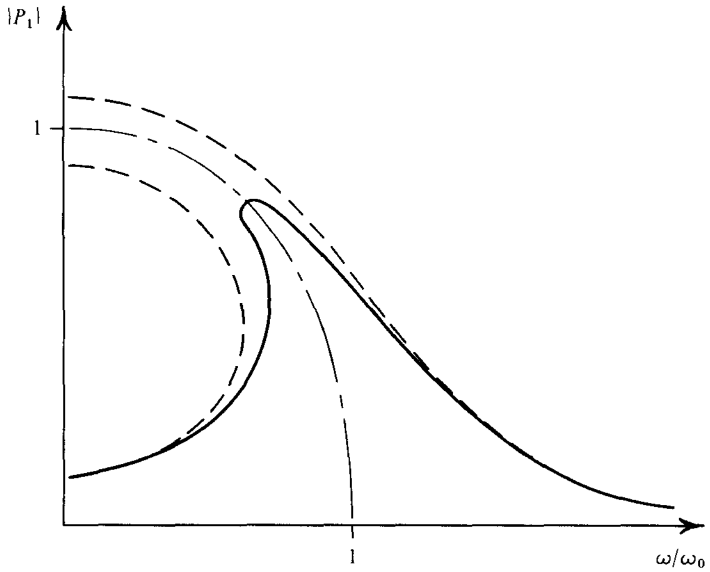

FIGURE 6. Response curves for $\beta = 0$ (– – –), $\beta > 0$ (——) and $Q_0 = 0$ (— — —). See equations (5.9) and (5.17).

in place of (2.6*a*, *b*). Invoking the transformation (3.1) and proceeding as in §3, we obtain

$$z/d = -p^2 + (\epsilon/2)^{\frac{1}{2}}(q+q_0), \tag{5.4}$$

$$\frac{dp}{d\theta} = q_0 - q - \beta\mathscr{X}p(1-p^2), \quad \frac{dq}{d\theta} = p(1-p^2) + (2\epsilon)^{\frac{1}{2}}pq_0, \tag{5.5a,b}$$

where

$$q_0 = (2\epsilon)^{-\frac{1}{2}}d^{-1}Z_0 \tag{5.6}$$

and $O(\epsilon)$ and $O(\beta\epsilon^{\frac{1}{2}})$ are neglected.

The principal effect of the term $(2\epsilon)^{\frac{1}{2}}pq_0$ in (5.5*b*) if

$$q_0 = Q_0 \cos \omega t, \tag{5.7}$$

as in the subsequent development, is to introduce even harmonics in the response. These could be significant if $\omega \simeq 2n\omega_0$ (in which case they could be calculated by an obvious extension of the following analysis) but otherwise are of little importance. Neglecting $(2\epsilon)^{\frac{1}{2}}pq_0$ and eliminating q between (5.5*a*, *b*), we obtain

$$\frac{d^2p}{d\theta^2} + \left(1 + \beta\frac{d}{d\theta}\mathscr{X}\right)p(1-p^2) = \frac{dq_0}{d\theta}. \tag{5.8}$$

Note that $\beta \ll 1$, which reflects the fact that the present model has a high Q if radiation provides the principal damping.

If q_0 is given by (5.7) and $O(\beta)$ is neglected, (5.8) is Duffing's equation for an undamped oscillator with a soft spring (Stoker 1950). The resulting response curves form a one-parameter (Q_0) family (see figure 6), the *spine* of which corresponds to the free oscillations ($Q_0 = 0$) analysed in §3.

Exact solutions of Duffing's equation are not known, and the usual analytical procedure is to expand p in a Fourier series and then to determine ω^2 and the amplitudes of the higher harmonics (odd only) as expansions in powers of the amplitude of the fundamental. It is expedient, in the present problem, to pose this Fourier series in the form

$$p = -(\omega/\omega_0)(P_1 \sin \omega t + P_3 \sin 3\omega t + \ldots). \tag{5.9}$$

Substituting (5.7) and (5.9) into (5.5a) and (5.8) and neglecting $O(\beta)$, we obtain

$$q = Q_0 \cos \omega t + (\omega/\omega_0)^2 (P_1 \cos \omega t + 3P_3 \cos 3\omega t + \ldots), \tag{5.10}$$

$$(\omega/\omega_0)^2 = (1 + \tfrac{3}{4}P_1^2)^{-1}\{1 - (Q_0/P_1)\} \tag{5.11}$$

and

$$P_3 = \tfrac{1}{4}(\omega/\omega_0)^2 \{(3\omega/\omega_0)^2 - 1\}^{-1} P_1^3, \tag{5.12}$$

wherein higher powers of P_1 are implicitly neglected. The next approximation includes $-(\omega/\omega_0)P_5 \sin 5\omega t$ and $5(\omega/\omega_0)^2 P_5 \cos 5\omega t$ in (5.9) and (5.10), respectively, and yields $P_5 = O(P_1^5)$, as well as improved approximations to $(\omega/\omega_0)^2$ and P_3.

The resistive component of $\mathscr{Z}$ (which represents radiation) becomes significant in the neighbourhood of resonance and may be approximated by

$$\mathscr{Z} = \tfrac{1}{2}(\omega/\omega_0) \tag{5.13}$$

if the external depth is approximated by d. (The right-hand side of (5.13) should be multiplied by d/d_E for an external depth d_E; see Miles & Lee (1975) and references given there regarding the effects of continental-shelf topography.) The reactive component of $\mathscr{Z}$ (which represents stored energy in an external neighbourhood of the mouth) has effects that are comparable with those of the kinetic energy in the basin and may be consistently neglected, as also may be the effect of damping on the harmonics (on the hypothesis that it is the fundamental that is being resonantly excited). The response then must be of the form [cf. (5.9)]

$$p = -(\omega/\omega_0) P_1 \sin(\omega t + \alpha) + \ldots, \tag{5.14}$$

where (by hypothesis) $\alpha = O(\beta)$. Substituting (5.7), (5.13) and (5.14) into (5.5a) and (5.8) and letting $\beta \downarrow 0$, we obtain

$$q = q_0 + (\omega/\omega_0)^2 [P_1 \cos(\omega t + \alpha) + \tfrac{1}{2}\beta P_1 \mathscr{P} \sin(\omega t + \alpha) + \ldots], \tag{5.15}$$

$$\alpha = -\tfrac{1}{2}\beta(\omega/\omega_0)^2 \{\mathscr{P} - (\omega/\omega_0)^2\}^{-1} \mathscr{P} \tag{5.16}$$

and

$$\{\mathscr{P} - (\omega/\omega_0)^2\}^2 + \tfrac{1}{4}\beta^2(\omega/\omega_0)^4 \mathscr{P}^2 = (Q_0/P_1)\{\mathscr{P} - (\omega/\omega_0)^2\}, \tag{5.17}$$

where

$$\mathscr{P} = 1 - \tfrac{3}{4}(\omega/\omega_0)^2 P_1^2. \tag{5.18}$$

The response curve described by (5.17) is sketched in figure 6. It is triple valued in a frequency domain $0 < \omega_1 < \omega < \omega_2 < \omega_0$ below the spine, which leads to hysteresis and jump phenomena as ω is varied through (ω_1, ω_2); see Stoker (1950, cha. 4, §4) for details.

Subharmonic resonance of the present model also is possible. The analysis follows that for Duffing's equation; see Stoker (1950, cha. 4, §§7 and 8).

6. Coupled basins

Let $Z_{1,2}$ be the free-surface displacements in a pair of basins of free-surface areas $S_{1,2}$ that are coupled by a canal of breadth b, length $2l$ and free-surface displacement z (figure 2). Continuity and the approximation of spatial uniformity then imply the constraint

$$S_1 Z_1 + S_2 Z_2 + 2blz = 0, \tag{6.1}$$

whilst convenient generalized co-ordinates are z and

$$Z = (S_2 Z_2 - S_1 Z_1)/(S_1 + S_2), \tag{6.2}$$

from which it follows that

$$Z_1 = -(Z + \epsilon z)/(1-\sigma), \quad Z_2 = (Z - \epsilon z)/(1+\sigma), \tag{6.3a, b}$$

where

$$\epsilon = bl/S, \quad S = \tfrac{1}{2}(S_1 + S_2), \quad \sigma = (S_2 - S_1)/(S_1 + S_2). \tag{6.4a, b, c}$$

The velocity in the canal $(-l < x < l)$, which must satisfy (2.1) and the boundary conditions [cf. (2.2)]

$$b(d+z)u = \begin{matrix} -S_1 \dot{Z}_1 \\ S_2 \dot{Z}_2 \end{matrix} \quad \left(x = \begin{matrix} -l \\ l \end{matrix}\right), \tag{6.5}$$

is given by [cf. (2.3)]

$$(d+z)u = (S/b)\dot{Z} - \dot{z}x. \tag{6.6}$$

The kinetic and potential energies implied by (6.3) and (6.6) are [cf. (2.4) and (2.5)]

$$T = (\rho l S^2/b)(d+z)^{-1}(\dot{Z}^2 + \tfrac{1}{3}\epsilon^2 \dot{z}^2) \tag{6.7}$$

and

$$V = (\rho l S^2/b)(\omega_\sigma^2/d)\{Z^2 + 2\epsilon\sigma Z z + \epsilon(1 - \sigma^2 + \epsilon) z^2\}, \tag{6.8}$$

where

$$\omega_\sigma^2 = (gbd/2l)(S_1^{-1} + S_2^{-1}) = \omega_0^2/(1-\sigma^2) \tag{6.9}$$

is the natural frequency (for small displacements). The counterparts of (3.1)–(3.4) are

$$Z = (2\epsilon)^{\frac{1}{2}}(1-\sigma^2)^{\frac{1}{2}} d q(\theta), \quad \dot{Z} = (2\epsilon)^{\frac{1}{2}}\omega_0 (d+z) p(\theta), \quad \theta = \omega_\sigma t, \tag{6.10a, b, c}$$

$$z/d = -p^2 - \delta q + O(\epsilon), \tag{6.11}$$

where

$$\delta = (2\epsilon)^{\frac{1}{2}}\sigma(1-\sigma^2)^{-\frac{1}{2}}, \tag{6.12}$$

$$H \equiv (T+V)/(4\rho g b d^2 l) = \tfrac{1}{2}(p^2 + q^2) - \tfrac{1}{4}p^4 - \tfrac{1}{2}\delta p^2 q + O(\epsilon), \tag{6.13}$$

$$dq/d\theta = (1-p^2)p - \delta p q, \quad dp/d\theta = -q + \tfrac{1}{2}\delta p^2, \tag{6.14a, b}$$

The solution of the phase-plane equations (6.14) for p is given by (3.8a, c, d) within $1 + O(\epsilon)$. The results (3.9), (3.11a), (3.12) and (3.13a) also hold in the present case, whilst (3.8b), (3.11b) and (3.13b) hold if $\delta = 0$. The solution for $\delta = 0$ $(S_1 = S_2)$ is given by

$$Z_{1,2} = Z_0(\mp \operatorname{cn}\phi \operatorname{dn}\phi + \epsilon^{\frac{1}{2}} k \operatorname{sn}^2\phi), \tag{6.15}$$

where the upper and lower signs correspond to Z_1 and Z_2, respectively, $Z_0 = (2\epsilon)^{\frac{1}{2}} dQ$ is the initial value of Z_2 (or of $-Z_1$), and ϕ and k are given by (3.8c, d). Note that $Z_1(t) = Z_2(t+\frac{1}{2}T)$, since shifting t by $\frac{1}{2}T$ is equivalent to shifting ϕ by $2K$, which changes the sign of $\operatorname{cn}\phi$ but not $\operatorname{dn}\phi$, so that the motions in the two basins are 180° out of phase, as might have been inferred from symmetry; on the other hand $Z_1 = -Z_2$ only at $t = \frac{1}{2}nT$ ($n = 0, 1, 2, \ldots$). It follows that the solution for the single basin is given by the corresponding solution for coupled basins of equal area if and only if $O(\epsilon^{\frac{1}{2}} k)$ is neglected, either because k is small, as in the linear approximation, or because $O(\epsilon^{\frac{1}{2}})$, rather than merely $O(\epsilon)$, is neglected compared with 1.

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